The Mathematics of the Longitude

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An academic exercise presented in partial fulfilment for the degree of Bachelor of Science with Honours in Mathematics.

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Contents

Acknowledgements

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Next, I would like to thank my family for their encouragements and support. Last but not least, I would like to thank Martin Yeo for being a good personal as well as spiritual companion to me.

Summary

The main objective of this project is to write a clear mathematical supplement to the book “Longitude” by Dava Sobel. This bestseller gives a wonderful summary of the history of the problem, but does not cover the mathematical background. In this project, a summary of the theory of navigation and the mathematical background of the different methods for finding the longitude are being covered.

Throughout history, it has been easy to determine latitude by measuring the angle between the Pole star or the sun at noon and the horizon by a simple angle measuring device or an advanced sextant. But how to determine longitude at sea had been a serious problem for a long time.

Since they could find latitude, but not longitude, navigators run mainly north and south until they reached the latitude of their destinations, and then turn east or west to make a landfall. Because the ships were not taking the most direct route to their destination, the journeys were long and dangerous. Thus, there was a need to know longitude.

In the ancient times, lunar eclipse was used as a clock for determining longitude. However, lunar eclipses are rare. Hence it is of no use to ships. Later, Jupiter's moon's eclipse was also used as a celestial clock for determining longitude. Unlike lunar eclipse, Jupiter's moon experienced a lot of eclipses. This idea actually worked but the difficulty in making these observations on a rocking ship made it impractical.

Lunar distance method was another theoretically possible method by measuring the angle between the moon and other celestial bodies to determine precise time. However, the calculations were difficult and time consuming. The heyday of Lunar distance method was probably from about 1780 to 1840.

Finally in 1759, a British clockmaker named John Harrison developed a clock, called the chronometer, which was consistently accurate at sea. Because the earth

rotates 15 degrees per hour, if you know the precise time at the Greenwich meridian of longitude, the difference between that time and the navigator's local time (the time at his location) will give him his longitude. At last the mariner had the ability to easily and consistently determine his exact position.

With the invention of the chronometer, the Lunar distance method was being displaced completely. The introduction of the chronometer not only solved the problem of measuring longitude but also made possible more flexible methods of fixing position which did not involve finding latitude and longitude separately. In 1837, Captain Charles Sumner devised a trigonometric method of obtaining from celestial observations the lines known as Sumner lines of position. In 1875, Frenchman Marcq St. Hilaire improved upon Sumner's trigonometric calculations.
Other methods such as the prime vertical method and the meridian transit method were also introduced for determining longitude.

Hence, with the chronometer, longitude can be determined using the any of those methods mentioned above. Thus, navigators no longer have to fear being lost at sea.

In this thesis, chapter 1 starts off with a short introduction to the problems of determining longitude. Next, chapter 2 to 4 introduce all the terminology and formulas used in this thesis. And chapter 5 gives a description on the evolution and the basic principal of a sextant. Then chapter 9 describes the invention of the chronometer. Finally, chapter 6 to 8 and chapter 10 to 13 touch on all those methods mentioned above to determine longitude in greater details.

Statement of Author's Contributions

In this project, I have tried with my best effort to write a clear mathematical supplement to the book “Longitude” by Dava Sobel. The summary of the theory of navigation and the mathematical background of the different methods for finding the longitude are being presented in such a way that you would find it easy to understand. Much work was done on the extraction and reorganization of basic ideas and terminology relevant to the context of this project (as shown from chapter 2 and 4). I have also furnished details and explanation of section 8.3.1 and 8.3.3. I have understood the Meridian Transit Method, the Sumner's Method and the Prime Vertical Method and presented the procedures of each of the methods in a systematic way with some guidance from my supervisor (Section 10.1, 11.4 and chapter 13).

Chapter 1

Introduction

Throughout history, it has been easy to determine latitude by measuring the angle between the pole star and the horizon by a simple measuring device or an advanced sextant. Later with the available of tables for the sun declination, latitude can also be determined by measuring the angle between the sun and the horizon.

At first sight it looks as though we ought to be able to find longitude in much the same way; we can't, because the earth is spinning. The essential difference is, of course, that latitude is measured with respect to the equator and poles of the earth which remain, stationary with respect to the stars or the sun. On the other hand, longitude is measured from some arbitrary north-south line, nowadays through Greenwich, and this line is not fixed with respect to the stars or sun but rotates with the earth. This means that to measure longitude by the stars or the sun we are concerned essentially with the rotation of the earth or, in other words, with the measurement of time.

As early as 1514, navigators knew well that the secret to determining longitude at sea lay in comparing the time aboard ship to the time at the home port at the very same moment. They could then convert the hour difference between the two places into a geographical one. Since the earth takes twenty-four hours to complete one full revolution of three hundred sixty degrees, in one hour it completes one twenty-fourth of that, or fifteen degrees. Each hour's time difference between the ship and the starting point therefore marks a progress of fifteen degrees of longitude to the east or west. Unfortunately, although navigators could figure out their local time at sea by watching the sun every day to see when it reached its highest point in the sky (at noon), they could not keep track of time at another place. For that they would have needed a clock or a watch set to the home port time. But pendulum clocks went haywire on the decks of rolling ships: they slowed down, or sped up, or stopped running altogether.

Early attempts to find longitude, such as the Lunar eclipses method, the Jupiter's moon eclipses method and the Lunar distances method were used at sea. But each of these methods, which will be discussed in later chapters, has it own weakness and was not practical at sea.

Astronomers tried to give mariners a way to tell time in two places at once by the moon and stars. Indeed, the great observatories in Paris and London were founded (in 1666 and 1674, respectively) not to conduct pure research in astronomy, but to perfect the art of navigation.

None the less, as the 18th century dawned no better method had come along and the increasing toll of lost ships and lives was causing growing concern. In a notorious accident in 1707, Royal Navy ships, believing themselves to be in deeper water further east, were wrecked on the Scilly Isles with the loss of almost 2000 lives.

Finally in 1714, the British Government offered, by Act of Parliament, 20 000 pounds for a solution which could provide longitude to within half-a-degree (2 minutes of time). The methods would be tested on a ship, sailing over the ocean, from Great Britain to any such Port in the West Indies as those Commissioners choose without losing their longitude beyond the limits before mentioned and should prove to be practicable and useful at sea.

Chapter 2

Terrestrial, Celestial and Horizon Coordinate Systems

2.1 Terrestrial Coordinate System

The position of an observer on the earth's surface can be specified by the terrestrial coordinates, **latitude** and **longitude**.

Lines of latitude are imaginary lines which run in an east-west direction around the world (Figure 2.1). They are also called parallels of latitude because they run parallel to each other. Latitude is measured in degrees (°).

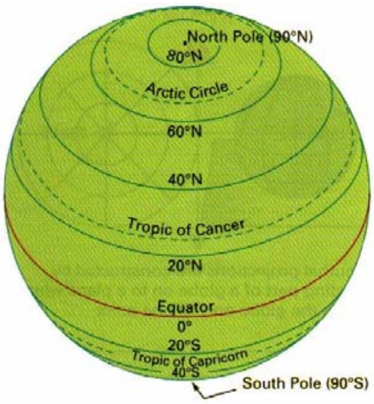

Figure 2.1 Lines of latitude.

The most important line of latitude is the **Equator** (0°). The North Pole is 90° North (90°N) and the South Pole is 90° South (90°S). All other lines of latitude are given a number between 0° and 90°, either North (N) or South (S) of the Equator. Some other important lines of latitude are the Tropic of Cancer (23.5°N), Tropic of Capricorn (23.5°S), Arctic Circle (66.5°N) and Antarctic Circle (66.5°S).

Lines of longitude are imaginary lines which run in a north-south direction, from the North Pole to the South Pole (Figure 2.2). They are also measured in degrees (°).

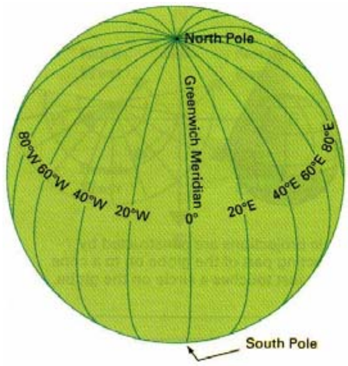

Figure 2.2 Lines of longitude.

Any circle on the surface of a sphere whose plane passes through the center of the sphere is called a **great circle**. Thus, a great circle is a circle with the greatest possible diameter on the surface of a sphere. Any circle on the surface of a sphere whose plane does not pass through the center of the sphere is called a **small circle**.

A **meridian** is a great circle going through the geographic poles, the poles where the axis of rotation (polar axis) intersects the earth's surface. The **upper branch** of a meridian is the half of the great circle from pole to pole passing through a given position; the **lower branch** is the opposite half. The equator is the only great circle whose plane is perpendicular to the polar axis. Further the equator is the only parallel of latitude being a great circle. Any other parallel of latitude is a small circle whose plane is parallel to the plane of the equator.

The **Greenwich meridian**, the meridian passing through the Royal Greenwich Observatory in London (closed in 1998), was adopted as the prime meridian at the International Meridian Conference in October 1884. Its upper branch (0°) is the reference for measuring longitudes, its lower branch (180°) is known as the **International Dateline**. All the lines of longitude are given a number between 0° and 180°, either East (E) or West (W) of the Greenwich Meridian.

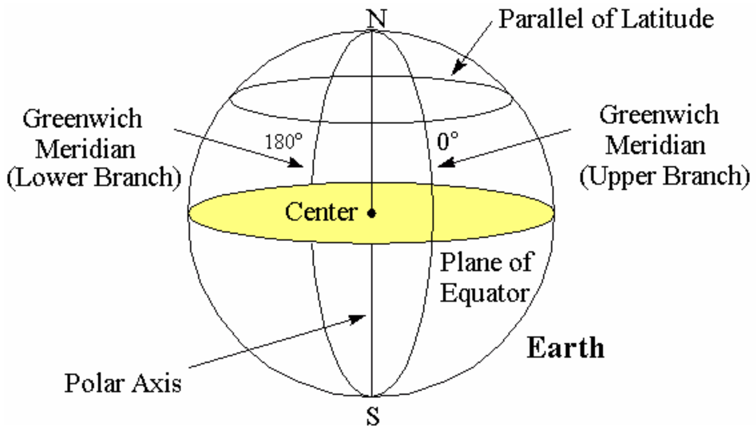

Figure 2.3

2.2 The Celestial Sphere

The **celestial sphere** is an imaginary sphere whose center coincides with the center of the Earth. It represents the entire sky; all celestial bodies other than the earth are imagined as being located on its inside surface. If the earth's axis is extended, the points where it intersects the celestial sphere are called the celestial poles; the **north celestial pole** is directly above the earth's north pole, and the **south celestial pole** is directly below the earth's south pole. The great circle on the celestial sphere halfway between the celestial poles is called the **celestial equator**; it can be thought of as the earth's equator projected onto the celestial sphere (Figure 2.4).

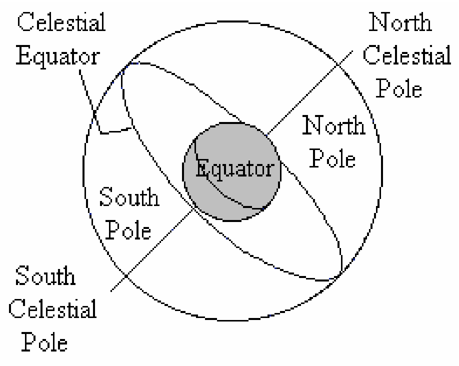

Figure 2.4

2.3 Other Reference Markers on the Celestial Sphere

The earth orbits the sun in a plane called the **ecliptic** (Figure 2.5). From our vantage point, however, it appears that the sun circle us once a year in that same plane. Hence, the ecliptic may be alternately defined as "the apparent path of the sun on the celestial sphere".

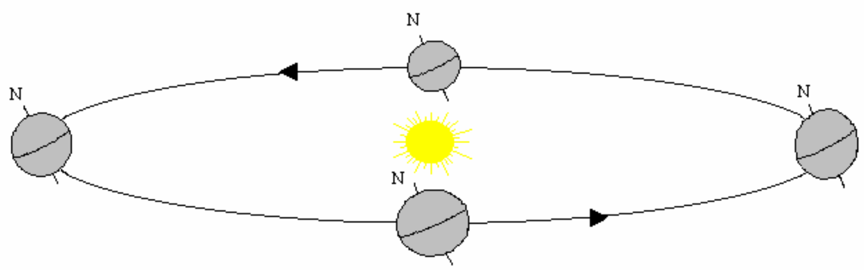

Figure 2.5 Earth orbits the sun in an ecliptic plane.

The ecliptic is tilted 23.5 degrees with respect to the celestial equator because the earth's rotation axis is tilted by 23.5 degrees with respect to its orbital plane. Be sure to keep distinct in your mind the difference between the slow drift of the sun along the ecliptic during the year and the fast motion of the rising and setting sun during a day.

The ecliptic and celestial equator intersect at two points: the **vernal (spring) equinox** and **autumnal (fall) equinox**. The sun crosses the celestial equator moving northward at the vernal equinox around March 21 and crosses the celestial equator moving southward at the autumnal equinox around September 22. When the sun is on the celestial equator at the equinoxes, everybody on the earth experiences 12 hours of daylight and 12 hours of night. The day of the vernal equinox marks the beginning of the three-month season of spring on our calendar and the day of the autumn equinox marks the beginning of the season of autumn (fall) on our calendar.

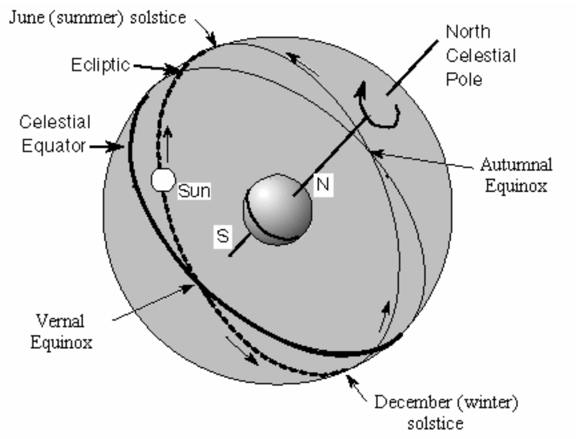

Figure 2.6

2.4 The Celestial Coordinate System

The celestial coordinate system is used for indicating the positions of celestial bodies on the celestial sphere.

To designate the position of a celestial body, consider an imaginary great circle passing through the celestial poles and through the body. This is the body's **hour circle**, analogous to a meridian of longitude on earth. Then measure along the celestial equator the angle between the vernal equinox and the body's hour circle. This angle is called the body's **right ascension**

(RA) and is measured in hours, minutes, and seconds rather than in the more familiar degrees, minutes and seconds. (There are 360 degrees or 24 hours in a full circle.) The right ascension is always measured eastward from the vernal equinox.

Next measure along the body's hour circle and the angle between the celestial equator and the position of the body. This angle is called the **declination (Dec)** of the body and is measured in degrees, minutes and seconds north or south of the celestial equator, analogous to latitude on the earth (Figure 2.7).

Right ascension and declination together determine the location of a body on the celestial sphere.

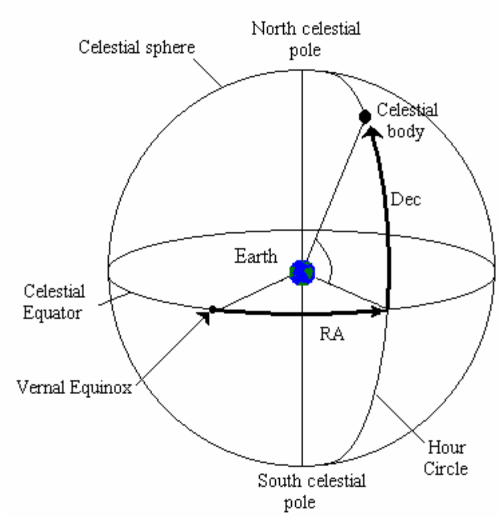

Figure 2.7 shows the Dec and RA of a celestial body.

2.5 Geographical Position of a Celestial Body

Now consider a line connecting the center of a celestial body and the center of the earth. The point where this line crosses the surface of the earth is called the **geographical position (GP)** of the body (Figure 2.8). An observer positioned in the GP of a body will see the body directly above his head.

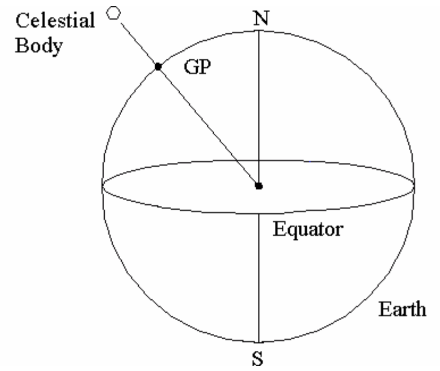

Figure 2.8 shows the GP of a celestial body.

Because both the equator and the celestial equator are in the same plane, the latitude of the GP is equal to the declination of the body. The longitude of the GP is called **Greenwich Hour Angle (GHA).** The GHA of any body is the angle, measured at the pole of the celestial sphere, between the Greenwich meridian and the hour circle of the body. The angle is measured along the celestial equator westward from the upper branch of the Greenwich celestial meridian, from 0° through 360° (Figure 2.9). The GHA differs from the longitude on the earth's surface in that longitude is measured east or west, from 0° through 180°, and remains constant. The GHA of a body, however, increases through each day as the earth rotates.

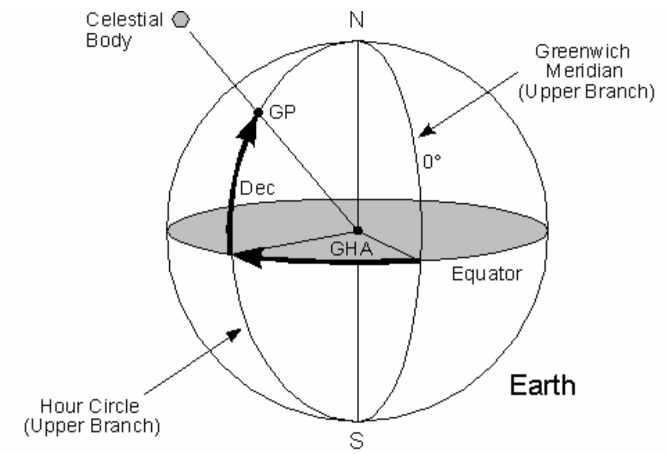

Figure 2.9 shows the Dec and GHA of a celestial body.

2.6 Horizon Coordinate System

The apparent position of a body in the sky is defined by the horizon coordinate system (Figure 2.10). The **altitude**, **H**, is the vertical angle between the horizontal plane to the line of sight to the body. The point directly overhead the observer is called the **zenith**. The **zenith distance**, **z**, is the angular distance between the zenith and the body. H and z are complementary angles (H + z = 90°). The **azimuth**, **Az_N**, is the horizontal direction of the body with respect to the geographic (true) north point on the horizon, measured clockwise through 360°.

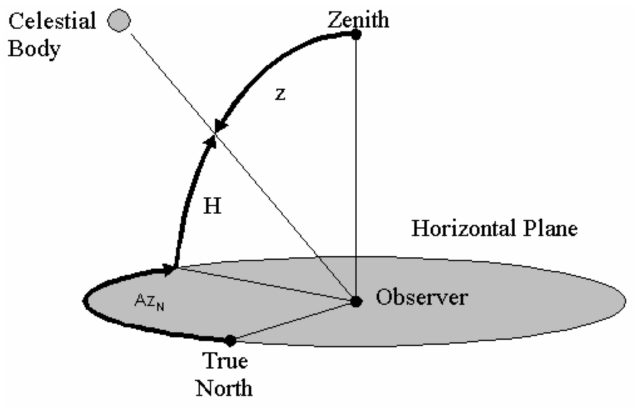

Figure 2.10

Each of the following imaginary horizontal planes parallel to each other can be used as the reference plane for the horizon coordinate system (Figure 2.11).

The **true horizon** is the horizontal plane tangent to the earth at the observer's position.

The **celestial horizon** is the horizontal plane passing through the center of the earth.

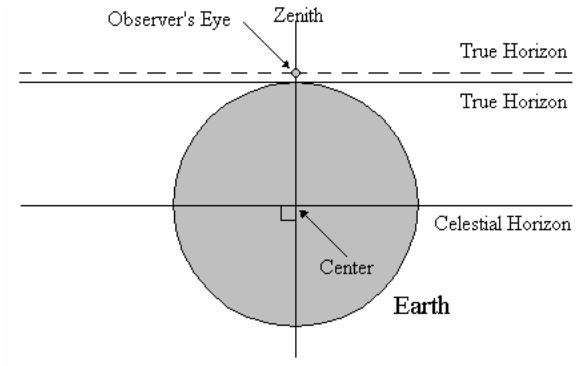

Figure 2.11

None of the above horizon coincides with the visible horizon used as the natural reference at sea. Only the altitude with respect to the celestial horizon is relevant to navigational calculations. Since it can not be measured directly, it has to be derived from the altitude with respect to the visible and true horizon (altitude corrections, see chapter 5).

Chapter 3

Time Measurement

3.1 Greenwich Mean Time

The time standard for celestial navigation is **Greenwich Mean Time**, **GMT**. GMT is based upon the GHA of the **mean sun** (an imaginary sun which move at a constant speed**):**

$$GMT[h] = \frac{GHA_{Mean\ Sun}[\circ]}{15} + 12.$$

(If GMT is greater than 24 h, subtract 24 hours.)

In other words, GMT is the angle, expressed in hours, between the lower branch of the Greenwich meridian and the hour circle through the mean sun (Figure 3.1). The GHA of the mean sun increases by exactly 15 degrees per hour, completing a 360 degrees cycle in 24 hours. Celestial coordinates tabulated in the Nautical Almanac refer to GMT.

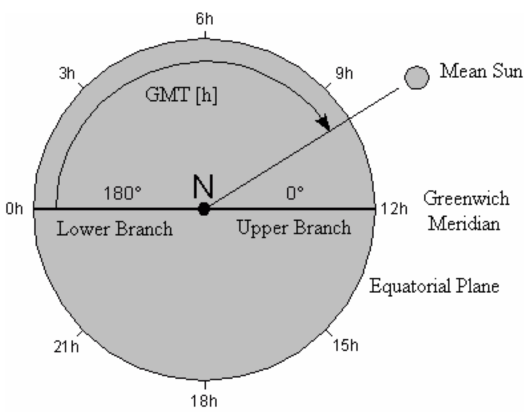

Figure 3.1 GHA of the mean sun.

3.2 Greenwich Apparent Time

The time based upon the GHA of the **apparent (observable) sun** is called **Greenwich Apparent Time**, **GAT**. The hourly increase of the GHA of the apparent sun is subject to periodic changes and is sometimes slightly greater, sometimes slightly smaller than 15 degrees during the course of a year. This behavior is caused by the eccentricity of the earth's orbit and its inclination to the plane of the equator.

3.3 Equation of Time

Most navigators are familiar with the fact that the sun transits the meridian on some days several minutes ahead of 12 noon local mean time and other days of the year it transits several minutes after 12 noon. That is, the mean sun and the apparent sun are out of step and the amount by which the apparent is ahead of the mean sun, in minutes and seconds of mean time, is known as the **equation of time**, denoted by **EoT**:

$$EoT = GAT - GMT.$$

EoT varies periodically between approximately -16 minutes and +16 minutes. Predicted values for EoT for each day of the year (at 0:00 and 12:00 GMT) are given in the Nautical Almanac.

3.4 Local Mean Time

Local mean time, **LMT**, is equal to the LHA of the mean sun as measured from the lower branch of the observer's meridian to the hour circle of the mean sun, that arc converted to time. Thus, if the observer were to locate at Greenwich then LMT would be identical to GMT.

3.5 Local Apparent Time

Local Apparent Time, **LAT**, follows the argument for LMT except that LHA is measured from the lower branch of the observer's meridian to the apparent sun. The difference between local mean time and local apparent time is also known as the equation of time.

Chapter 4

Spherical Trigonometry

4.1 The Spherical Triangle

A spherical triangle is formed by three planes passing through the surface of a sphere and through the sphere's center. In other words, a spherical triangle is part of the surface of a sphere, and the sides are not straight lines but arcs of great circles (Figure 4.1).

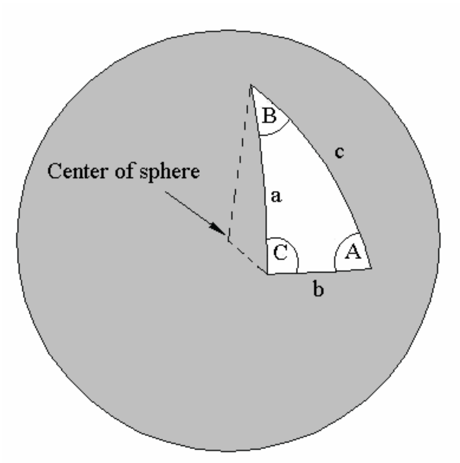

Figure 4.1 A spherical triangle on the surface of a sphere.

Any side of a spherical triangle can be regarded as an angle - the angular distance between the adjacent vertices, measured from the center of the sphere. For example, in Figure 4.1, length of side a is equal to the angle it subtends at the center. And angle of A is equal to the angle between 2 planes. The interrelations between angles and sides of a spherical triangle are described by the **law of sines**, the **law of cosines for sides** and the **law of cosines for angles**.

4.2 Spherical Trigonometry

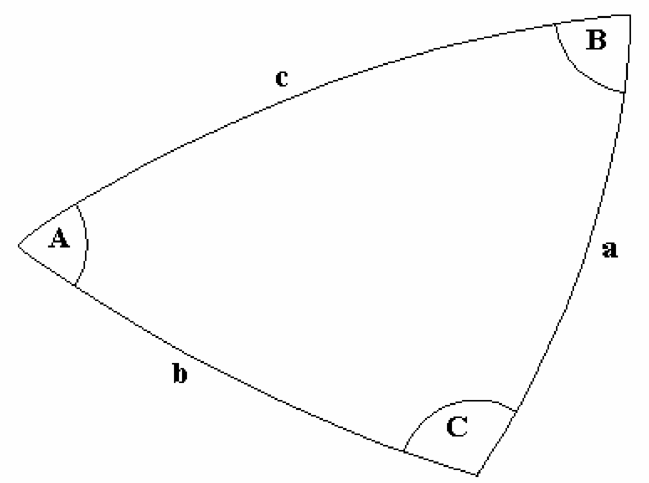

Figure 4.2 A spherical triangle.

Law of sines:

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)}$$

Law of cosines for sides:

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A)$$
$$\cos(b) = \cos(a)\cos(c) + \sin(a)\sin(c)\cos(B)$$
$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$$

Law of cosines for angles:

$$\cos(\mathrm{A}) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cos(a)$$
$$\cos(B) = -\cos(A)\cos(C) + \sin(A)\sin(C)\cos(b)$$
$$\cos(C) = -\cos(A)\cos(B) + \sin(A)\sin(B)\cos(c)$$

The proof of these formulas can be found in navigational textbooks.

These formulas allow us to calculate any quantity (angle or side) of a spherical triangle if three other quantities are known. In particular, the law of cosines for sides is of interest for navigational purposes.

Chapter 5

How to Measure the Altitude of a Celestial Body

5.1 Introduction

The astrolabe invented by the Greeks was the first instrument used to measure altitude of a celestial body. The astrolabe was mostly used in the Medieval Ages and by the Islamic world. However, it was cumbersome because it had to be suspended and often produced large, unavoidable errors in rough weather.

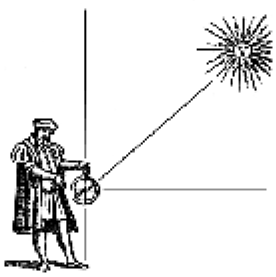

Figure 5.1 A man using the astrolabe.

The development of the cross-staff supplanted the use of the astrolabe. The cross-staff consisted of a long piece of wood with a crosspiece that could slide along the staff. The navigator would sight a celestial body along the top of the crosspiece and move it so that the bottom tip met the horizon. Navigators often used the sun as a sight, and the cross-staff often caused eye damage for longtime navigators.

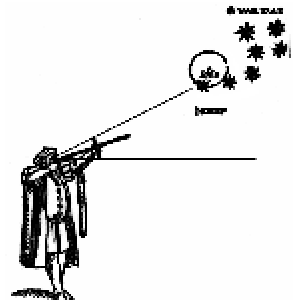

Figure 5.2 A man using the cross-staff.

The back-staff was then developed in 1590 so that navigator didn't have to look into the sun. The navigator faced away from the sun and used the shadow cast by the device to measure the angle between the body and the horizon. Later, smoked mirrors were added so that bodies which were too dim to give a shadow could be used while at the same time, dimming the reflection of the sun so that it could still be used. This foreshadowed the modern sextant.

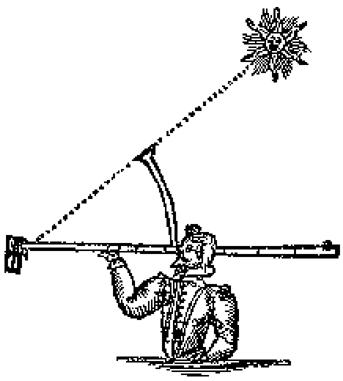

Figure 5.3 A man using the back-staff.

The modern **sextant** was invented by Sir Isaac Newton in 1700. However, he never built one. It was an Englishman named John Hadley and an American named Thomas Godfrey who built two different versions of the sextant, both equally useful. Later, Paul Vernier added a second graduated arc to make measurements more precise. Even later, the micrometer screw were invented, which further added to the precision of the instrument and became what we now recognize as the modern sextant.

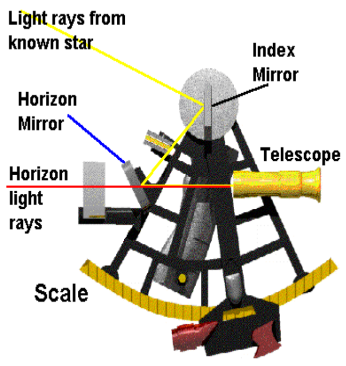

Figure 5.4 A picture of a sextant.

5.2 The Sextant

The **sextant** is an instrument used for measuring the altitude of a celestial body above the sea horizon. Figure 5.5 is a diagram embodying the more important features of the instrument. The sextant consists of a system of two mirrors, a telescope and a graduated arc. The index mirror I is mounted on the index arm IP. The index mirror I and the index arm IP can rotate about an axis (perpendicular to the plane of paper) at I. To any given position of the index mirror and of the index arm IP, there is a corresponding reading on the graduated arc. The horizon mirror H is fixed to the framework of the instrument and perpendicular to the plane of the paper. T is a small telescope attached to the framework.

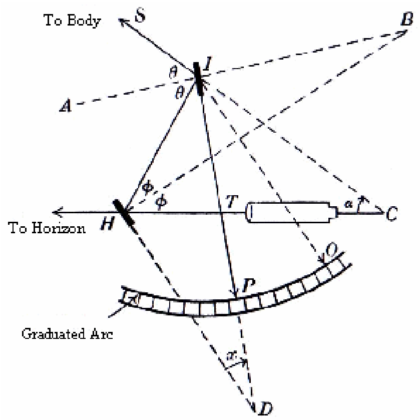

Figure 5.5 Important features of a sextant.

To find the altitude of a celestial body above the sea horizon, the observer holds the instrument in a vertical plane and points the telescope so as to see the horizon. He then moves the index mirror I by means of the arm IP until the image of the body is observed to be in his field of view. When the image appears on the line of the horizon (Figure 5.6), he notes the reading on the graduated arc.

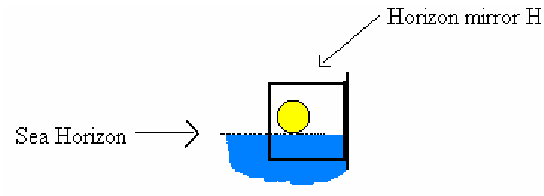

Figure 5.6 The sun's image is on the line of the sea horizon.

Let IS denote the direction of a celestial body. A ray in the direction SI is reflected by the index mirror I along IH; it is then reflected by the horizon mirror H along HT, and the body is thus observed in the telescope in the direction in which the sea horizon is seen.

The altitude of the body is simply related to the inclination of the index mirror I to the horizon mirror H - in Figure 5.5 the inclination is the angle IDH, which we denote by *x*. Let AIB and HB be the normals to the mirrors I and H; then the angle IBH is evidently *x*. The laws of reflection give

$$S\hat{I}A = A\hat{I}H = \theta$$

and $$I\hat{H}B = B\hat{H}C = \phi.$$

If a is the body's altitude above the sea horizon, then $S\hat{C}H = a$. From the triangle IHC, the exterior angle $S\hat{I}H = I\hat{C}H + I\hat{H}C$, so that

$$2\theta = 2\phi + a \,. \qquad (1)$$

Similarly, from the triangle IBH,

$$\theta = \phi + x . \qquad (2)$$

Hence from (1) and (2),

$$a = 2x , \qquad (3)$$

or the body's altitude is twice the angle between the mirrors I and H (or between their normals). The altitude a is zero when x is zero, that is when the mirrors I and H are parallel. In Figure 5.5, IO is parallel to the fixed direction HD; O is the zero-point of the scale. The angle OIP can thus be found from the reading on the graduated arc and, by (3), the body's altitude is twice this angle.

The arc of the sextant is generally about one-sixth of the circumference of a circle (hence the name "sextant"), but instead of having 60 divisions each representing one degree, the arc is divided into 120 divisions. In this way, the altitude is read directly from the scale without the necessity of applying the factor 2 of equation (3). With the aid of sub-divisions and a vernier, altitude can be read with a first-class instrument to one-tenth of a minute of arc.

5.3 The Vernier Scale

In figure 5.7, 10 divisions of the so-called vernier scale V equal 9 divisions of the main scale S. It follows that the interval between divisions on V equals 0.9 times the interval between divisions on S. When the zero points of V and S are aligned, as shown in Figure 5.7 (A), the 1 mark on V will lie to the right of the 1 mark on S by 0.1 times the main-scale interval; similarly, the 2 mark on V will lie to the right of the 2 mark on S by 0.2 times the main-scale interval, and so on. Suppose now the main scale S is moved to the right of the fixed vernier V by an arbitrary distance, say, 0.6 times the main-scale interval as shown in Figure 5.7 (B). The zero of the vernier will lie between the zero and 1 marks on the main scale and, further, the 6 mark on V will be aligned with the 6 mark on S. In other words, the numerical value of the vernier mark which coincides with a mark on the main scale indicates the decimal part to be added to the main-scale reading of the sextant.

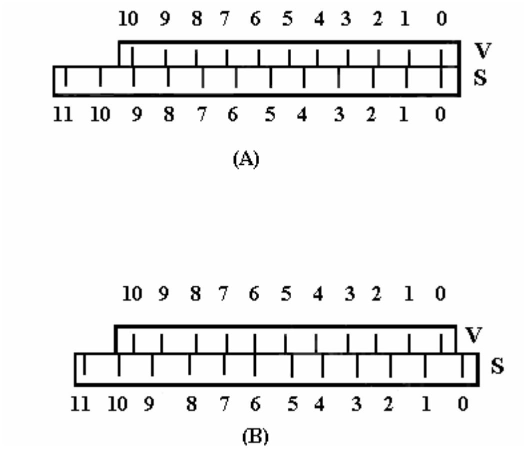

Figure 5.7 Vernier scale: showing subdivisions of a main scale.

5.4 Sextant Corrections

The **sextant altitude**, **Hs**, is the altitude as indicated by the sextant before any corrections have been applied. Hs contains systematic errors and can only be used for navigational calculations after several corrections have been applied.

(a) Index error

Index error, **IE**, is a constant error due to a lack of parallel alignment between the index and horizon mirrors. A sextant, unless recently calibrated, usually has an index error that has to be subtracted from the reading before they can be processed further.

To check for index error, hold the sextant in your right hand and look at the sea horizon. By moving the index arm, line up the real and mirror horizons so that both appear as a single straight line. Now, look at the scale. If it reads zero, there is no index error. If the scale reads anything but zero, there is an index error. For example, if the scale reads +5’ when the horizons are aligned, then 5’ is subtracted from Hs; if the reading is below the zero mark, for example –5’, then –5’ is subtracted from Hs.

1^{st} correction: H1 = Hs - IE

(b) Dip

Due to the curvature of the earth's surface, the apparent or visible sea horizon usually appears several arcminutes below the true horizon, depending on the **height of eye**, **HE** (Figure 5.8). The difference between the apparent or visible sea horizon and the true horizon is known as **dip**. Dip depends on the height of the observer's eyes. The higher the observer is from the sea horizon, the greater is the dip.

Since the sextant sights on the visible sea horizon and not on the true horizon, dip corrections must always be made on sextant observations. A table of dip corrections (height by dip correction) can be found in the Nautical Almanac. Such corrections are negative. Hence they are always subtracted.

The altitude obtained after applying corrections for index error and dip is referred to as **apparent altitude**, **Ha**. The apparent altitude is measured with respect to the true horizon.

2^{rd} correction: Ha = H1 - Dip

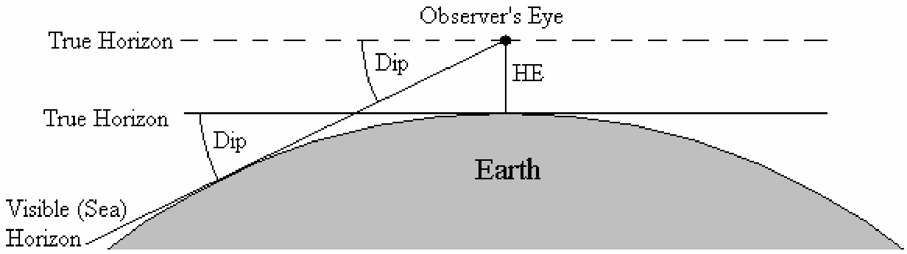

Figure 5.8 The visible sea horizon is dipped below the true horizon.

(c) Refraction

As light passes from a less to a more dense medium, i.e. from space to earth, it is deflected toward the earth. This phenomenon is called refraction. Since the eye cannot detect the curvature of the light ray, an observed body appears to be higher in the sky. The angle **R** is the **angle of refraction** and represents the angular distance between the apparent and true position of the body (Figure 5.9). Refraction depends on the altitude of the observed body. For example, if the observed body is at the zenith, there is no refraction. And as the altitude of observed body decrease, refraction will increase.

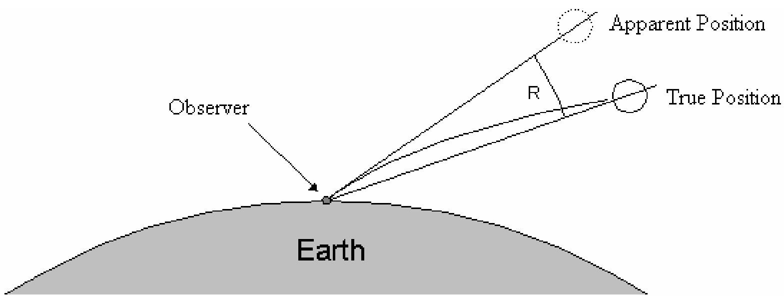

Figure 5.9 The apparent position of the celestial body is appeared to be higher than it's true position.

The nautical navigator altitude correction tables for refraction can be found in the Nautical Almanac. There is one table for stars and planets, another for the sun (with different values for sighting the upper and lower limbs), and another for the moon. Correction for refraction is always subtracted.

3^{rd} correction: H3 = Ha - R

(d) Parallax

Calculations of celestial navigation are measured with respect to the celestial horizon. However, the altitude Ha is measured with respect to the true horizon. Hence, correction has to be made. Such difference in angle between the altitude measured at the center of the earth (H4) and that measured at the surface of the earth (H3) is known as **Parallax**, **PA** (Figure 5.10). Parallax is significant only when the object is near to the earth. Hence, in navigational practice, only the parallax of the moon need to be taken into account. Since PA is the difference between H3 and H4, therefore as a correction, PA must be added to H3 to get H4.

$$4^{th} \text{ correction:} \qquad H4 = H3 + PA$$

From Figure 5.10, we can see that PA will change as the moon moves from the horizon to the zenith. Towards the zenith the value of PA diminishes toward zero. At the true horizon, the value of PA will be at the maximum. Parallax of the moon on the true horizon is known as **horizontal parallax**, **HP**. It is the value recorded in the Nautical Almanac and serves as the basis for computing the parallax correction for the moon.

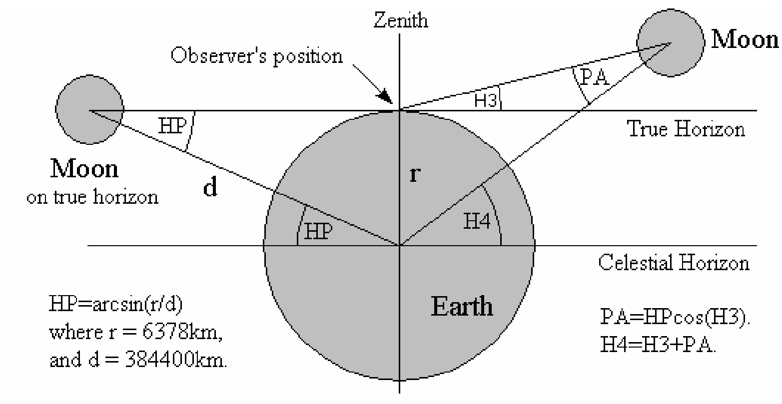

Figure 5.10 Horizontal parallax and parallax of the moon.

(e) Semi-diameter

Semi-diameter, **SD**, is the angular distance between center and limb of a body (Figure 5.11). When observing the sun or the moon, it is not possible to locate the center with sufficient accuracy. It is therefore common practice to measure the altitude of the lower or upper limb of the body. However, it is the center of the body that serves as reference for its positional coordinates. Consequently, if we sight on the lower limb of the sun or the moon we must add the semi-diameter to our sextant reading. And if the upper limb is observed, the semi-diameter value is subtracted.

Semi-diameter of sun and moon vary slightly from day to day depending on their distances from the earth. The correction tables for the sun, in the Nautical Almanac, include semi-diameter and refraction. The differences in the corrections for upper and lower limbs are due to the differential effects of refraction on the two limbs. The corrections for the moon, in the Nautical Almanac, include dip, refraction, semi-diameter, and parallax. Semi-diameter corrections for other bodies are insignificant and can be ignored.

5^{th} correction: $H5 = H4 \pm SD$

The altitude obtained after applying the above corrections (in the above sequence), suitable to be used for navigational calculations, is called **observed altitude**, **Ho**. That is Ho=H5.

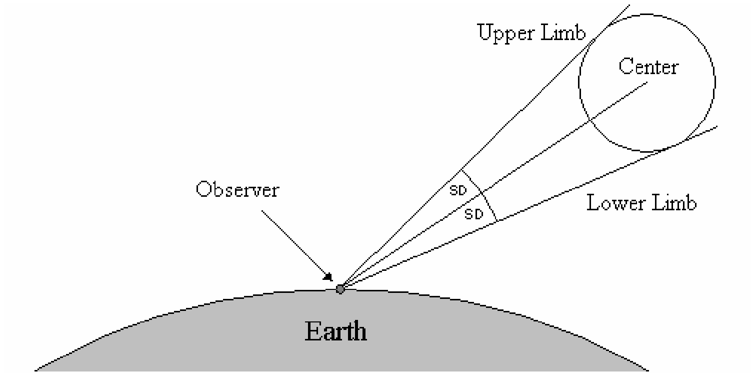

Figure 5.11 Semi-diameter of a celestial body.

Now with the observed altitude, latitude is known. However, the longitude is not known yet. In the next few chapters, we shall see how to determine longitude.

Chapter 6

Lunar Eclipse Method

6.1 Introduction

In the olden days, lunar eclipses have often been interpreted as bad omens. A well-known historical example dates back to Christopher Columbus. On his fourth voyage to America in 1504, Columbus faced problems. His ships were in poor condition due to shipworms and the vessels had to be beached in Jamaica. A lot of his supplies were stolen. Half of his crew had mutinied and worst still, the indigenous population refused to supply them with food.

In this dire situation, Columbus had a great idea. Western European astronomers had calculated that a lunar eclipse would occur during the night of September 14-15, 1409. Columbus announced to the indigenous population that, due to the wrath of the Gods, the moon would disappear during the following night (Figure 6.1). The lunar eclipse appeared right on schedule and Columbus finally got his food supplies.

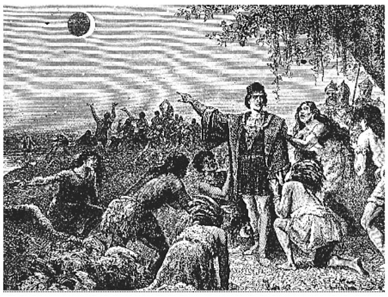

Figure 6.1 Columbus announced to the indigenous population that the moon would disappear during the following night.

6.2 What is a Lunar Eclipse

A **lunar eclipse** occurs when the moon is covered by the shadow of the earth. That is, the earth comes between the sun and the moon, casting the earth's shadow on the surface of the moon (Figure 6.2).

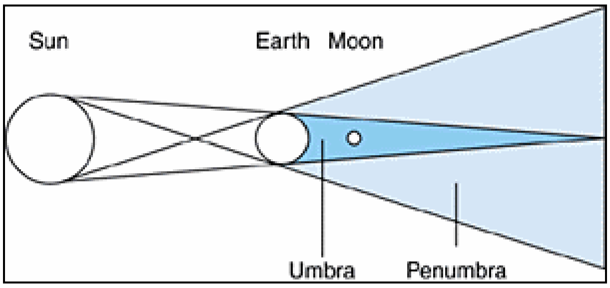

Figure 6.2 A lunar eclipse occurs when the moon is covered by the shadow of the earth.

A partial lunar eclipse happens when the moon only passed through the outer part of the shadow (the penumbra). At this moment, the observer on the earth would see the moon only partially dimmed (Figure 6.3).

On the other hand, a total lunar eclipse happens when the moon passes through the umbra. During this time, the observer on earth will see that the entire moon is covered (Figure 6.4).

Figure 6.3 A partial lunar eclipse.

Figure 6.4 A total lunar eclipse.

6.3 Lunar Eclipse Method

Being lost at sea was one of the greatest fears of the early navigator. While the latitude of the navigator position can be easily determined with a sextant, the longitude is harder to determine. As mentioned earlier, longitude is a matter of time. There is no fixed point of reference in the sky, but only the uniform rotation of the earth every day. Thus if one knows the difference in local time at two points, one knows the longitudinal distance between them. Measuring the time where one is located is relatively straightforward: by observing the sun or the stars in the sky. Knowing the local time elsewhere at that same moment is another matter.

In the ancient times, the only practical method for determining longitude was the well-known lunar eclipse method. Lunar eclipse provides an easily observable event and everyone sees the eclipse happen at the same time. Thus, Lunar eclipse provides a method of time "synchronization". The procedures of the lunar eclipse method are relatively simple. First, we determine the local time that the lunar eclipse starts or ends by direct observation. Next either we obtain the local time of another place from tables of Moon or we ask someone to station at a distant place to take the time where the lunar eclipse starts or ends again. Then we compare the local time for that event against the local time at that distant place. The difference in the two times is the difference in longitude.

6.4Weakness of the Lunar Eclipse Method

Although this method to find the navigators' longitude is simple and reliable, the use of eclipses was obviously of little use to them. This is because lunar eclipses are relatively rare. There are only about two lunar eclipses per year.

In ancient times, where accurate clocks were not available, sandglass was used as a timekeeper. As the sandglass does not provide accurate time, the timing of the lunar eclipse is inaccurate. The result is in an error in the longitude obtained. It was only in the 18th century that an accurate chronometer was invented.

Moreover, tables of the Moon used in the olden days were imperfect. And a prediction of a lunar eclipse from them could easily be off by a quarter of an hour. Such an error, added to errors in determine the local time of the eclipse, could result in an error in longitude of 8 degrees.

If tables of the Moon are not available, then one has to rely on someone at a distant place to take the time where lunar eclipse starts or ends again. And then one has to wait until the information is passed to one. Thus, this method is impractical at sea.

Chapter 7

Eclipses of Jupiter's Satellites Method

7.1 Introduction

Although the lunar eclipse method was simple and reliable and was used extensively on land, the use of eclipses was of little use to ships as eclipses are rare. The lack of an effective method for determining longitude had serious consequences for navigators and their vessels, often laden with precious goods. With increasing importance of navigation and a series of maritime disasters due to uncertainty about longitude, there was an urgent need to search for a satisfactory method of determining longitude at sea.

In 1610, Galileo Galilei (1564-1642) discovered, with his telescope, that four satellites surrounded Jupiter. Every now and then the satellites would pass in and out of the shadow of Jupiter and Galileo realised that these eclipses could provide a celestial clock in the determination of longitude. His discovery was extremely important because it revealed the existence of celestial bodies which rotated around a body other than the earth.

Figure 7.1 Galileo Galilei (1564-1642).

7.2 Eclipses of Jupiter's Satellites Method

Jupiter casts a shadow, which stretches behind it in space, and eclipses occur when a satellite passes into the planet's shadow. When a satellite moves into eclipse, its light dims until the satellite disappears from view. A few hours later it emerges from the other side of the shadow, and as it does so it brightens and become visible again.

The procedures of the eclipses of Jupiter's Satellites method for determining longitude are relatively simple. First, we use a telescope to watch one of the satellites of Jupiter as it revolves around Jupiter. Secondly, we note the local time at which this satellite appears or disappears behind Jupiter. Thirdly, we obtain the time from tables at which it appears to do so at Greenwich. Then the difference between these two times gives us our longitude.

7.3Disadvantages of the Eclipses of Jupiter's Satellite Method

Using the eclipses of Jupiter's satellite method to determine longitude requires the use of a telescope. Galilean telescopes, in use before about 1650, were not optimum for observation of Jupiter's satellites. Their fields of view were extremely small, so that it was difficult to find the planet and its satellites and almost as difficult to keep it in the field of view. In addition, the telescopes could not be used during overcast weather at night because the satellites are not visible.

Besides telescope, this method also requires the use of tables of the times these satellites eclipsed. However, accurate tables were not available. It was not until 1668, that the first reasonably accurate tables were published by Giovanni Domenico Cassini (1625 - 1712).

Figure 7.2 Giovanni Domenico Cassini (1625-1712).

The eclipses of Jupiter's satellites provided a wonderful method of establishing longitude on land, but it was practically useless in sea. This is because making observation with a telescope at the deck of a rocking ship was impossible. Galileo made some trials of telescope attached to a helmet (which he called "celatone") on ships, but this approach only worked with rather low-powered telescopes. Hence, Galileo's effort to make telescopes adequate for observing Jupiter's satellites was not successful.

Chapter 8

Lunar Distance Method

8.1 Introduction

Though the Jupiter's satellites eclipse method proved to be a better method than the lunar eclipse method, it was still not a successful method to be used at sea. This is because navigators were unable to observe the satellites with a telescope on the rocking deck of a ship. Hence, there was a desperate need to discover a practical method of determining longitude at sea.

In 1475, a well-known German astronomer Regiomontanus suggested the lunar distance method to find longitude. Lunar distance method is the measurement of the angular distance between the moon and a star or between the moon and the sun. While his theory was sound, the 15th century did not possess the instrument for measurement the lunar distance and the lunar tables to make it work. In was not until mid 15th century when the cross-staff was proposed for making lunar distance measurements (Figure 8.1).

Figure 8.1 Lunar distance from a star is being measured with a cross-staff.

Even with the availability of instruments and tables, the lunar distance method was still not successful. There were two reasons for this. Firstly, there was no instrument with which the angular distance between the moon and a star could be measured with sufficient accuracy from the deck of a moving ship. Secondly, astronomers could not predict the relative positions of the moon and the star with the necessary precision. It was only with the development of the sextant and the available of better lunar tables, that the lunar distance method for determining longitude proved successful.

8.2Lunar Distance Method

A lunar distance measurement is most easily made with the sextant when the angular distance between the Moon and the other body is not great. At nighttime, measurement is made between the Moon and one of the 57 selected stars listed in the daily pages of the Nautical Almanac. In daytime, the Sun may be used as the second body.

The procedures of the lunar distance method are described below.

(1) Take three simultaneous sets of lunar observations. Each set of lunar observations consists of a lunar distance, the altitude of the Moon, the altitude of the sun or star and the local mean time (LMT) of the observations.

All these required angles should be measured simultaneously and repeatedly so that at least 3 sets of lunar observations are obtained.

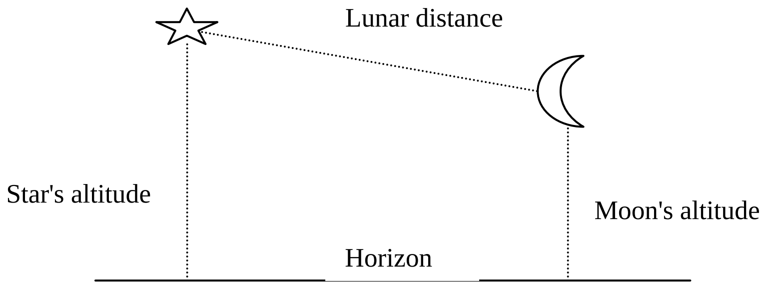

Figure 8.2 A Lunar distance observation.

(2) Calculate the mean of measured angles and LMT of the observations made. That is to say, the sums of the lunar distances, the moon's altitudes, the sun or the star's altitude, and the LMT should be divided by the number of sets. By so doing, small errors of observation are eliminated or reduced.

(3) Use LMT and EoT, obtained from the almanac, to find the Local Apparent Time (LAT).

$$LAT = LMT + EoT$$

(4) Obtain the True Lunar Distance by clearing the mean of the measured lunar distances in (1) of the effects of refraction and parallax (Section 8.3). It is at this moment that we need the altitude of the moon and the altitude of the stars or the sun in order to ascertain the exact values of the refraction and parallax for the moon and sun or refraction for the stars.

(5) Find Greenwich Apparent Time (GAT) from the True Lunar Distance in (4) by interpolation in the lunar-distance tables in the almanac.

(6) The LONGITUDE is the difference between Local Apparent Time in (3) and Greenwich Apparent Time in (5).

8.3 Clearing the Lunar Distance

When finding longitude by the lunar method, the most tedious part of the process is that which is known as clearing the distance. This involves reducing the apparent lunar distance to the true lunar distance or, in other words, clearing the apparent distance from the effects of parallax and refraction in order to ascertain the angle at the Earth's center between the directions of the moon's center and the star, or sun's center. In the following, two methods for clearing the lunar distance are discussed. One is **Borda's Method** and the other one is **Merrifield's Method**. Borda's Method allows us to get an exact true lunar distance. Merrifield's Method, on the other hand, allows us to get an approximated true lunar distance.

8.3.1 Borda's Method for Clearing the Distance

Chevalier Jean Borda, the French mathematician and astronomer, was born at Dax in 1733. He entered the French Navy and made nautical astronomical investigation for which he became well known. In 1787 Borda published his method of clearing lunar distances. At the time of its introduction to navigators, and for many decades afterwards, Borda's method was considered by competent authorities to be the best.

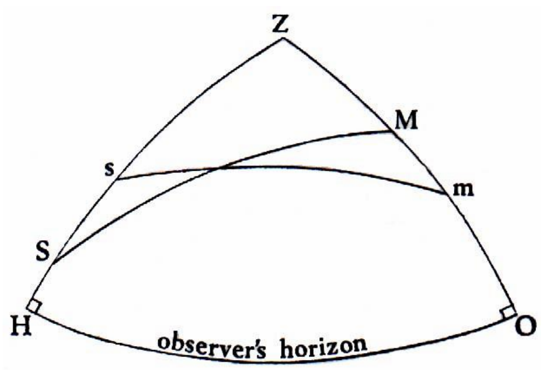

Figure 8.3 A navigational triangle.

Let *M* be the true altitude of the Moon (arc *MO*),

M be the apparent altitude of the Moon (arc *Om*),

S be the true altitude of the Sun or star (arc *SH*),

s be the observed altitude of the Sun or star (arc *sH*),

D be the true lunar distance *(arc SM)*,

d be the observed lunar distance *(arc sm)*.

Due to the parallax of the Moon, the observed altitude of the Moon is smaller than the true altitude of the Moon as shown in Figure 8.3. And due to refraction, observed altitude of the Sun or star is bigger than the true altitude of the Sun or star.

Using triangle ZSM and the law of cosines for sides, we have

$$\cos(Z) = \frac{\cos(SM) - \cos(ZS)\cos(ZM)}{\sin(ZS)\sin(ZM)} \, ,$$

$$\Downarrow$$

$$\cos(Z) = \frac{\cos(SM) - \sin(HS)\sin(OM)}{\cos(HS)\cos(OM)} \, . \qquad (1)$$

Using triangle Zsm and the law of cosines for sides, we have

$$\cos(Z) = \frac{\cos(sm) - \cos(Zs)\cos(Zm)}{\sin(Zs)\sin(Zm)} \, ,$$

$$\Downarrow$$

$$\cos(Z) = \frac{\cos(sm) - \sin(Hs)\sin(Om)}{\cos(Hs)\cos(Om)} \, . \qquad (2)$$

Equating (1) and (2), we have

$$\frac{\cos(D)-\sin(S)\sin(M)}{\cos(S)\cos(M)}=\frac{\cos(d)-\sin(s)\sin(m)}{\cos(s)\cos(m)},$$

$$1+\frac{\cos(D)-\sin(S)\sin(M)}{\cos(S)\cos(M)}=1+\frac{\cos(d)-\sin(s)\sin(m)}{\cos(s)\cos(m)},$$

$$\frac{\cos(S)\cos(M)+\cos(D)-\sin(S)\sin(M)}{\cos(S)\cos(M)}=\frac{\cos(s)\cos(m)+\cos(d)-\sin(s)\sin(m)}{\cos(s)\cos(m)},$$

$$\frac{\cos(D)+\cos(M+S)}{\cos(S)\cos(M)}=\frac{\cos(d)+\cos(m+s)}{\cos(s)\cos(m)},$$

$$\frac{1-2\sin^2\left(D/2\right)+2\cos^2\left\{(M+S)/2\right\}-1}{\cos(S)\cos(M)}=\frac{2\cos\left\{(m+s+d)/2\right\}\cos\left\{(m+s-d)/2\right\}}{\cos(s)\cos(m)},$$

$$\sin^2\left(D/2\right)=\cos^2\left\{(M+S)/2\right\}-\frac{\cos(S)\cos(M)\cos\left\{(m+s+d)/2\right\}\cos\left\{(m+s-d)/2\right\}}{\cos(s)\cos(m)}.$$

Let

$$\cos^2(\theta)=\frac{\cos(S)\cos(M)\cos\left\{(m+s+d)/2\right\}\cos\left\{(m+s-d)/2\right\}}{\cos(s)\cos(m)}. \qquad (3)$$

Then, $\sin^2\left(D/2\right)=\cos^2\left\{(M+S)/2\right\}-\cos^2(\theta)$

$$=\frac{1}{2}\{1+\cos(M+S)\}-\frac{1}{2}\{1+\cos 2\theta\}$$

$$=\frac{1}{2}\{\cos(M+S)-\cos(2\theta)\}$$

$$=\sin\left\{(M+S)/2+\theta\right\}\sin\left\{\theta-(M+S)/2\right\}$$

$$\therefore \sin\left(D/2\right) = \left[\sin\left\{(M+S)/2 + \theta\right\} \sin\left\{\theta - (M+S)/2\right\}\right]^{\frac{1}{2}}. \qquad (4)$$

Apply logarithm to equations (3) and (4), we have

$$\log\cos(\theta) = \frac{1}{2}\left\{\log\sec(m) + \log\sec(s) + \log\cos\left\{(m+s+d)/2\right\}\right\}$$
$$+\frac{1}{2}\left\{\log\cos\left\{(m+s-d)/2\right\} + \log\cos(M) + \log\cos(S)\right\},$$

$$\log\sin\left(D/2\right) = \frac{1}{2}\left[\log\sin\left\{(M+S)/2 + \theta\right\} + \log\sin\left\{\theta - (M+S)/2\right\}\right].$$

8.3.2 Application of Borda's method

Observed Altitudes	True Altitudes	Observed Distance
m = 13°29'27"	M = 14°18'32"	d = 107°52'04"
s = 31°11'34"	S = 31°10'07"	

Using the above data and apply Borda's method to get the true lunar distance.

d	107°52'04"		
m	13°29'27"	log sec(m)	0.012151°
s	31°11' 34"	log sec(s)	0.067815°
d+m+s	152°33'05"		
(d+m+s)/2	76°16'32.5"	log cos(m+s+d)/2	-0.62479°
(m+s-d)/2	-31°35'31.5"	log cos(m+s-d)/2	-0.06966°
M	14°18'32"	log cos(M)	-0.013686°
S	31°10'07"	log cos(S)	-0.067705°

$(M+S)/2$ $\quad \underline{22°44'19.5}''$

$$\log\cos\theta = 1/2\{\ 0.012151 + 0.067815 - 0.62479 - 0.06966$$
$$- 0.013686 - 0.067705\ \}$$
$$= -0.3479375$$

Hence $\theta = 63.33264127°$

$= 63°19'57.51''$

$\theta+(M+S)/2$	86°04'17.01"	$\log\sin\{\theta+(M+S)/2\}$	-0.0010°
$\theta-(M+S)/2$	40°35'38.01"	$\log\sin\{\theta-(M+S)/2\}$	-0.1866°

$$\log\sin(D/2) = 1/2\{\ -0.0010 - 0.1866\ \}$$
$$= -0.0938$$

$\therefore\ D = 107.3653855°$

$= 107°21'55.3''$

Hence, the true lunar distance is 107°21'55.3".

8.3.3 Merrifield's Approximation Method for Clearing the Lunar Distance

Dr John Merrifield was a well-known author of works on navigation. He invented the Merrifield method for clearing lunar distances. This method is direct in its application, requires no special tables, and is claimed to be a very close approximation well adapted for sea use.

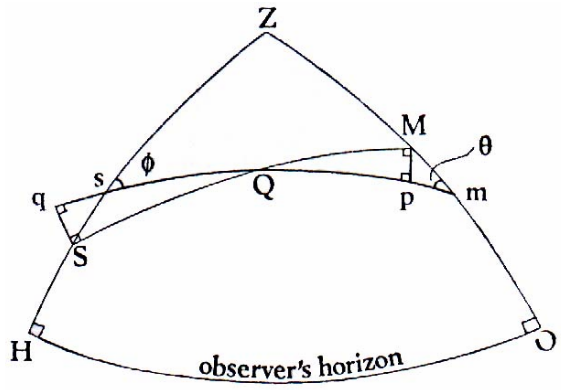

Figure 8.4 **A navigational triangle.**

Let Z be the zenith of an observer,

d be the observed lunar distance (arc sm),

D be the true lunar distance (arc SM),

m be the apparent Moon position,

s be the apparent second body position,

y be the Moon's apparent zenith distance (arc Zm),

z be the second body's apparent zenith distance (arc Zs),

$S' = (y+z+d)/2$,

$S^* = (m+s+d)/2$,

C = Moon's correction for altitude (arc Mm),

c = second body's correction for altitude (arc sS),

$\theta = Zms$,

$\phi = Zsm$.

This method is an approximation method because the true lunar distance D (arc MS) is approximated by arc qp.

Thus

$$
\begin{aligned}
D &= arc\ MS \\
&= arc\ qp \\
&= sm - pm + sq \\
&= d - C\cos(\theta) + c\cos(\phi) \\
&= d - C\left[1 - 2\sin^2\left(\theta/2\right)\right] + c\left[1 - 2\sin^2\left(\phi/2\right)\right] \\
&= d - (C - c) + 2\left[C\sin^2\left(\theta/2\right) - c\sin^2\left(\phi/2\right)\right]. \qquad (1)
\end{aligned}
$$

Now, $\sin^2\left(\theta/2\right) = \dfrac{1-\cos(\theta)}{2}$

$$= \frac{1}{2} - \frac{1}{2}\cos(\theta)$$

$$= \frac{1}{2} - \frac{1}{2}\left[\frac{\cos(z) - \cos(d)\cos(y)}{\sin(y)\sin(d)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(d)\sin(y) - \cos(z) + \cos(d)\cos(y)}{\sin(y)\sin(d)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos(y-d) - \cos(z)}{\sin(y)\sin(d)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos(y-d) - \cos(2S'-d-y)}{\sin(y)\sin(d)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos\{(S'-d)-(S'-y)\} - \cos\{(S'-d)+(S'-y)\}}{\sin(y)\sin(d)}\right]$$

$$= \frac{\sin(S'-d)\sin(S'-y)}{\sin(y)\sin(d)}$$

$$= \frac{\sin\left[(y+z-d)/2\right]\sin\left[(z+d-y)/2\right]}{\sin(d)\sin(y)}$$

$$= \frac{\sin\left[90 - (m+s+d)/2\right]\sin\left[(m+d-s)/2\right]}{\sin(d)\cos(m)}$$

$$= \frac{\cos(S^*)\sin(S^*-s)}{\sin(d)\cos(m)}.$$

Similarly, $\sin^2\left(\phi/2\right) = \dfrac{\cos(S^*)\sin(S^*-m)}{\sin(d)\cos(s)}$.

Hence,

$$C\sin^2\left(\theta/2\right) - c\sin^2\left(\phi/2\right)$$

$$= \frac{C\cos\left(S^*\right)\sin\left(S^* - s\right)}{\sin(d)\cos(m)} - \frac{c\cos\left(S^*\right)\sin\left(S^* - m\right)}{\sin(d)\cos(s)}$$

$$= \frac{\cos\left(S^*\right)}{\sin(d)}\left[\frac{C\sin\left(S^* - s\right)}{\cos(m)} - \frac{c\sin\left(S^* - m\right)}{\cos(s)}\right]$$

$$= \cos\left(S^*\right)\cos ec(d)\left[C\sin\left(S^* - s\right)\sec(m) - c\sin\left(S^* - m\right)\sec(s)\right]$$

$$= (M - N)\cos ec(d)\cos\left(S^*\right),$$

where $M = C\sec(m)\sin\left(S^* - s\right)$,

and $N = c\sec(s)\sin\left(S^* - m\right)$.

Then from formula (1), we have

$$D = d - (C - c) + 2(M - N)\cos ec(d)\cos\left(S^*\right).$$

8.3.4 Application of Merrifield's Approximation Method

Observed Altitudes	True Altitudes	Observed Distance
m = 13°29'27"	M = 14°18'32"	d = 107°52'04"
s = 31°11'34"	S = 31°10'07"	

Using the above data and Merrifield's approximation method to obtain the approximated true lunar distance.

$C = M - m = 49'05''$
$c = s - S = 01'27''$

m	$13°29'27''$	$\sec(m)$	$1.028375707°$
s	$31°11'34''$	$\sec(s)$	$1.169003337°$
d	$107°52'04''$	$\cos ec(d)$	$1.050677202°$
$m+s+d$	$152°33'05''$		

S^*	$76°16'32.5''$	$\cos(S^*)$	$0.237250267°$
$S^* - s$	$45°04'58.5''$	$\sin(S^* - s)$	$0.708129343°$
$S^* - m$	$62°47'05.5''$	$\sin(S^* - m)$	$0.889295566°$

$C - c \quad 47'38''$

$$M = C\sec(m)\sin(S^* - s)$$
$$= 35'44.62''$$

$$N = c\sec(s)\sin(S^* - m)$$
$$= 01'30.44''$$

$$M - N = 34'14.18''$$

$$\therefore D = d - (C - c) + 2(M - N)\cos ec(d)\cos(S^*)$$
$$= 107°21'30.1''$$

Hence, the approximated true lunar distance is 107°21'30.1" .

By comparing with the exact lunar distance obtained in section 8.3.2, we see that the approximated true distance obtained above is very close to our exact value of 107°21'55.3".

Chapter 9

Chronometer

It had always been realized that in principle the use of a transportable clock was the simplest of all methods of finding longitude but in practice the mechanical problems of developing a suitable clock was complicated. The first practical timepiece which was sufficiently accurate to find longitude when carried in a ship was made in Yorkshire by John Harrison. When the offer of the great prize for longitude was made in 1714, Harrison was 17 years old and he apparently devoted his life to winning it.

Figure 9.1 John Harrison (1693-1776)

The technical problem which Harrison solved was to develop a clockwork mechanism which would run at a uniform rate in a moving ship under conditions of varying temperature. He eventually cracked this problem with his fourth timepiece (H4) which he completed in 1759 at the age of 66. Harrison made it look like an ordinary pocket watch although it was laid on a cushion in a box whose level could be adjusted by hand.

Figure 9.2 H4, Harrison's fourth marine chronometer.

The introduction of the marine chronometer not only solved the problem of measuring longitude but also made possible more flexible methods of fixing position which did not involve finding latitude and longitude separately. The best example is Sumner's method, which will be discussed in a later chapter. There are later refinements of this way of fixing position but they are mostly based on Sumner's method.

Chapter 10

Meridian Transit Method

10.1 Meridian Transit Method

We know that the earth rotates with an angular velocity of 15 degrees per hour with respect to the mean sun. Using this fact, the time of local meridian transit of the (true) sun, $T_{Transit}$, can be used to calculate the observer's longitude. Note that $T_{Transit}$ is measured as GMT. The procedures of the Meridian Transit method are described below.

(1) Determine the $T_{Transit}$ [h]. (Refer section 10.2)

(2) From the Nautical Almanac, for $T_{Transit}$, find $EoT_{Transit}$ by interpolation.

(3) Use GMT and $EoT_{Transit}$ to find GAT.

$$GAT[h] = GMT[h] - EoT[h]$$

(4) Use GAT and LAT to determine longitude. Note that our LAT is 1200h.

$$Lon[°]=15(LAT-GAT)$$

10.2 How to Determine $T_{Transit}$

Local Apparent Noon (**LAN**) is the local time when the (true) sun is at our meridian. That is the LAT is 1200 noon. A sextant reading is taken at a time before the LAN (T1). Assuming that the sun moves along a symmetric arc in the sky and keeping our sextant reading, the time (T2) past LAN at which the sun has an altitude equal to our sextant reading is taken. Then the mean of T_1 and T_2 is our $T_{Transit}$.

$$i.e.\ T_{Transit} = \frac{T_1 + T_2}{2}$$

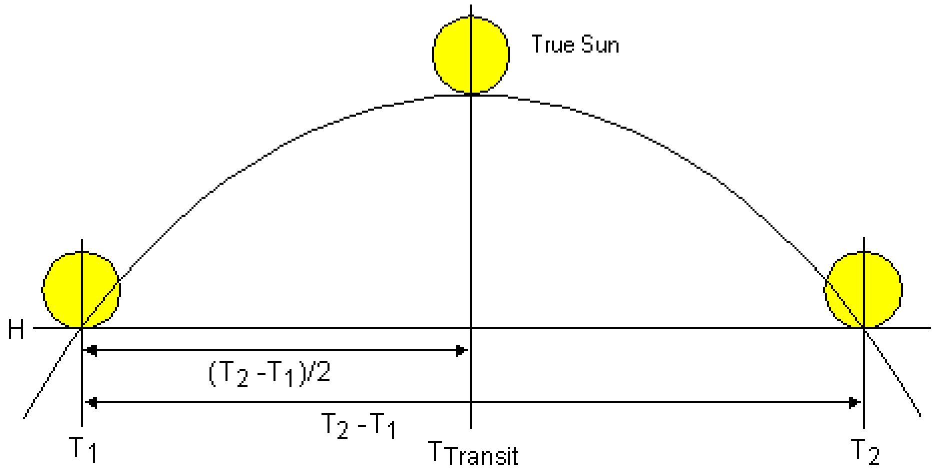

Figure 10.1 Sun moves along a symmetric arc in the sky.

10.3 Errors of Meridian Transit Method

Unfortunately, the arc of the true sun is usually not quite symmetrical with respect to $T_{Transit}$ due to the changing declination of the sun. As a result, $T_{Transit}$ is not exactly the mean of T_1 and T_2.

The resulting systematic error in longitude is negligible around the times of the solstices when Dec is almost constant, and is greatest at the times of the equinoxes when the rate of change of Dec is greatest (Figure 10.2).

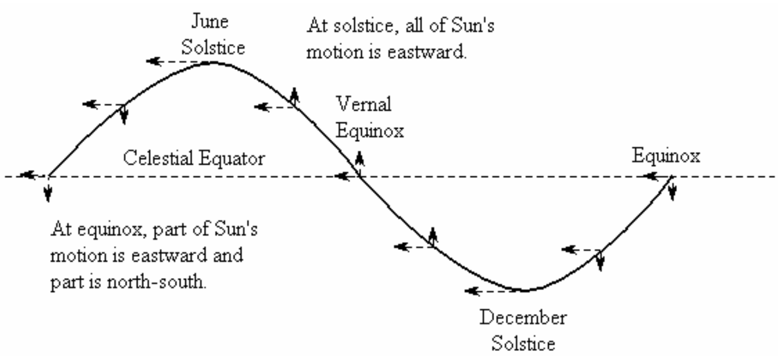

Figure 10.2 Motion of the Sun.

Chapter 11

Sumner's Method

11.1 Introduction

Sumner lines were discovered by chance by a sea captain, Charles Sumner in 1837. On a voyage back home, he was worried because he had been sailing for several days in bad weather, and had not been able to see the sun or any stars, the coast was getting near, and he did not know where he was. Suddenly there was a break in the clouds, so he grabbed his sextant and snatched a quick sun sight, before the clouds covered the sky again. He also timed his observation with a chronometer.

Now he was wondering what to do with this information so he played a "what if" game. He knew the altitude formula and he said, "what if my latitude is…" and calculated the corresponding longitude and he plotted it on a chart. After doing that half a dozen times he suddenly realized that all the points he was marking on the chart seemed to fall on a straight line. Without thinking about it any more, he saw that the line needed pushing north by 30 odd miles to lead straight into his destination, so he turned north for 30 miles, then turned to port until he was sailing parallel to that Sumner line. His crew were a bit perplexed at that, wondering if the captain had gone mad, but when suddenly they arrived right at the entrance of the harbor, they thought he was a genius. And so did the rest of the sailing community.

Although this single observation did not tell him the position of his ship, Captain Sumner realized that it did tell him that the ship must lie somewhere on a line, actually a **position circle**, which he could draw on his chart - a remarkable discovery to be made by the captain of a ship during a storm.

11.2 The Position Circle

Let S be the heavenly body,

U be the GP of the heavenly body,

C be the center of the earth,

PGQ be the Greenwich meridian,

P and Q be the north and south pole respectively.

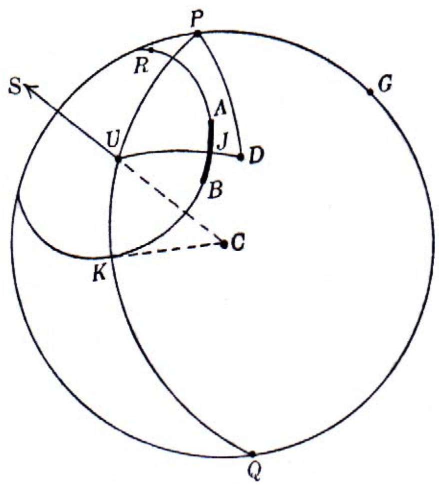

Figure 11.1 A position circle.

Now, consider the observation of altitude from which the true zenith distance z is derived. If the observer were actually at the point K on the earth's surface, then angle KCU or the great circle arc UK is the true zenith distance z. Since the GMT at which the observation of altitude was made is noted, the GP of the body U can then be found from the Nautical Almanac. Thus U is a definite point on the earth's surface. The true zenith distance z is also known, and therefore the observer must be situated somewhere on a small circle KJR of which U is the pole, every point of this small circle being an angular distance z from U. This small circle is called the **position circle**.

It is noted that a single observation of a heavenly body leads only to a certain small circle, on which the observer is situated. If we suppose that a similar observation of another heavenly body is made at the same GMT, a second position circle will be derived on which the observer must be situated. Therefore his actual position must be at one or other of the two points of intersection of the two position circles. As the approximated position of the ship is always known, there is no difficulty in deciding which of the two points is the correct position.

11.3 Sumner Lines of Position

Sumner's method makes use of two points on the position circle to draw a straight line, called the **sumner line of position (**Sumner LoP**)**. Thus, Sumner's line of position is a **secant**. As a sumner line tells us that we can be any where on this line, it does not tells us our position. So in order to obtain our position, we need to another sumner's line. Then the intersection of the two sumner lines is our position.

11.4 Sumner's Method

Although Sumner's method is quite old, it is an interesting sight reduction alternative. The procedures of the Sumner's method are described below.

(1) Choose a convenient position on your map, preferable the point where two grid lines intersect. This position is called assumed position, AP.

(2) Find the altitude **H** of a selected celestial body by sextant and note the **GMT**.

(3) From the Nautical Almanac, for the date and GMT, find the **GHA** and **Dec** of the body.

(4) Use H, Lat_{AP} and Dec to compute **LHA**.

$$LHA = \arccos\left[\frac{\sin(H) - \sin(Lat_{AP})\sin(Dec)}{\cos(Lat_{AP})\cos(Dec)}\right]$$

(5) From our assumed latitude, Lat_{AP}, two longitudes are obtained based on the following formulas:

Lon1 = *LHA* - *GHA*

If *Lon*1<-180°, set *Lon1* = *Lon*1 + 360°.

*Lon*2 = 360° - *LHA* - *GHA*

If *Lon*2<-180°, set *Lon*2 = *Lon*2 + 360°.

If *Lon*2>180°, set *Lon*2 = *Lon*2 - 360°.

(6) From the longitudes obtained in (5), choose the one that is nearer to our assumed longitude, Lon_{AP} and mark it on the chosen parallel of latitude.

(7) Repeat the procedure with another assumed position to obtain another point.

(8) Draw a straight line through both obtained points. And this is your approximate Sumner LoP.

(9) A second Sumner LoP is obtained by entering the data of a second body.

(10) The point of intersection of both LoPs is your position (fix).

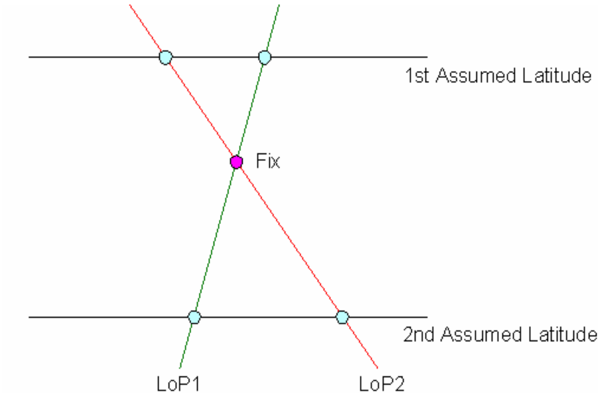

Figure 11.2

Chapter 12

The Intercept Method

12.1 Introduction

The practice of position fixing by celestial bodies has been used by navigators in ocean passage until accurate and low cost electronic navigation aids, like transit satellite and GPS, are available. Sun sights during the daylight hours and star sights during twilight are obtained to determine the ship’s position. The Marcq Saint-Hilaire or **intercept method** is the most popular way to reduce a sight for **a position line**. The intercept method uses the difference between the observed true altitude and the calculated altitude of the ship's assumed position to give an intercept for plotting the position line on chart.

12.2 The Position Line

As we have seen, the altitude observation of a heavenly body yields the information that at the time of observation the ship is situated on a certain small circle KJR (Figure 11.1). We shall suppose that in Figure 11.1, D represents the approximated position of the ship at this moment. It is clear that the only part of the position circle with which the navigators need concern is that part in the immediate neighborhood of D. Their objects then are to represent on the chart this part AJB of the position circle (shown with a heavy line in Figure 11.1).

Now the latitude and longitude of U and of D are known, hence the length of the great circle arc UD can be calculated. But this arc is simply the zenith distance of the hypothetical observer situated at D. We shall call this zenith distance UD the calculated zenith distance. Now the length of the arc UJ is known from the observation, it is the true zenith distance z; hence, by subtraction we obtain the length of the arc DJ. This arc DJ is known as the **intercept**. The arc DJ is perpendicular to the position circle at J, for U is the pole of KJR.

Also, the spherical angle UDP is easily seen to be the azimuth of the heavenly body, at the GMT concerned, for an observation at D can be calculated. In Figure 11.1, the azimuth of J is the same as the azimuth of U, which we now suppose to be known. Thus, under the circumstances depicted in figure 11.1, where the calculated zenith distance UD is greater than the true zenith distance UJ, the navigator can draw from the approximated position on his chart a straight line in the direction given by the azimuth of the heavenly body. He then marks off along this line a distance equal to the intercept, and through the point so obtained he draws a straight line, called the **position line**, perpendicular to the line of azimuth. The position line represents on his chart the portion AJB of the position circle KJR (Figure 11.1). This line is know as a **tangent** to the position circle at J. It is clear that if D is within the position circle, that is, if the calculated zenith distance is less than the true zenith distance, the intercept will be marked off in the direction opposite to that given by the azimuth.

12.3 The Intercept Method

The intercept method of obtaining a position line was discovered by a French navigator Marcq St. Hilaire in 1875. Now with an idea of what the position line is, we can now described the procedures of applying the intercept method to obtain ship current position, called "fix". The procedures of obtaining a fix are described below.

(1) Choose a convenient position on your map, preferably the point where two grid lines intersect. This position is called assumed position, **AP**.

(2) Find the altitude **H** of a selected celestial body by sextant and note the **GMT**.

(3) From the Nautical Almanac, for the date and GMT, find the **GHA** and **Dec** of the body.

(4) Use the GHA and Lon_{AP} to compute **LHA**.

$$LHA = GHA \pm Lon_{AP}$$

+ : For eastern longitude

- : For western longitude

Lon_{AP} : Longitude of AP.

If LHA is not in the range between 0° and 360°, add or subtract 360°.

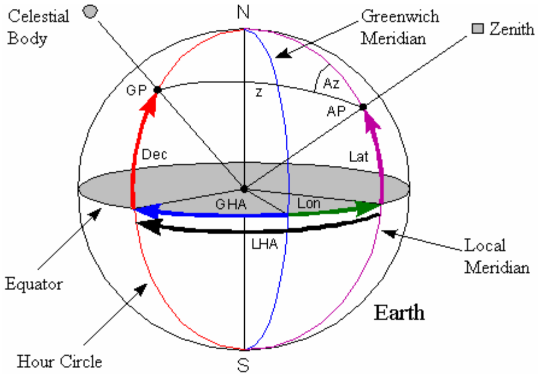

Figure 12.2

(5) Calculate the altitude of the observed body as it would appear from AP. This is called calculated or computed altitude, **Hc**.

$$Hc = \arcsin\left[\sin\left(Lat_{AP}\right)\sin(Dec) + \cos\left(Lat_{AP}\right)\cos(Dec)\cos(LHA)\right],$$

where Lat_{AP} is the geographic latitude of AP.

(6) Calculate the **azimuth** of the body, $\mathbf{Az_N}$, the direction of GP with reference to the geographic north point on the horizon, measured clockwise from 0° through 360° at AP

$$Az = \arccos\left[\frac{\sin(Dec) - \sin(Hc)\sin\left(Lat_{AP}\right)}{\cos(Hc)\cos\left(Lat_{AP}\right)}\right].$$

The azimuth angle, Az, the angle formed by the meridian going through AP and the great circle going through AP and GP, is not necessarily identical with Az_N since the arccos function yields results between 0° and +180°. To obtain Az_N, apply the following rules:

$$Az_N = \begin{cases} Az & if \ \sin LHA < 0 \\ 360° - Az & if \ \sin LHA > 0. \end{cases}$$

(7) Calculate the **intercept**, **Ic**, the difference between observed (Ho) and computed (Hc) altitude. For the following procedures, the intercept is expressed in distance units:

$$Ic[nm] = 60(Ho - Hc) \quad \text{or} \quad Ic[km] = \frac{40031.6}{360}(Ho - Hc),$$

The mean perimeter of the earth is 40031.6 km.

(8) Take the map and draw a suitable segment of the azimuth line through AP. Measure the intercept, Ic, along the azimuth line (towards GP if Ic>0, away from GP if Ic<0) and draw a perpendicular line through the point thus located. This perpendicular line is the plotted (approximate) position line.

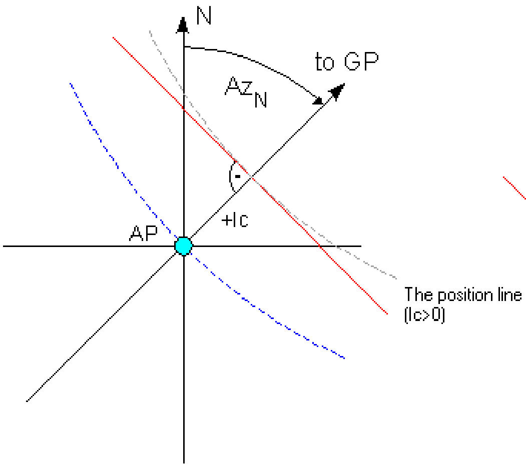

Figure 12.3a

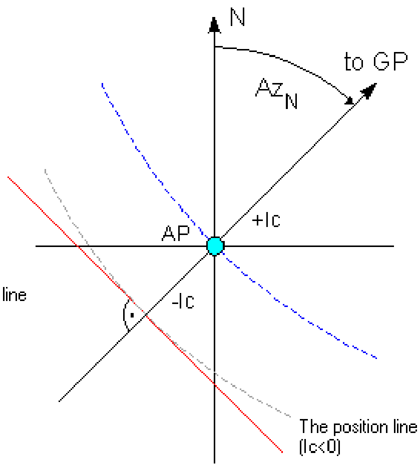

Figure 12.3b

(9) To obtain the second position line needed to mark your position, repeat the above procedure (same AP) with altitude and GP of another celestial body or the same body at a different time of observation. The point where both plotted position lines intersect is your established position (fix).

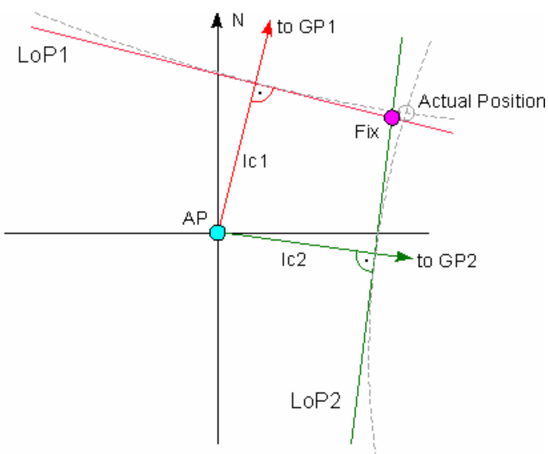

Figure 12.4

If a high precision is required, the intercept method can be applied **iteratively**, that is repeating the calculation with the obtained position (fix) instead of AP (same altitudes and GP of the 2 observed bodies), until the obtained position remains virtually constant.

Chapter 13

Prime Vertical Method

Before the introduction of the Sumner's method, **prime vertical method** is a common method used by navigator to find their longitude. At sea, navigator will determine the latitude at noon. After sometimes, **dead reckoning** is used to determine the new local latitude. Then this obtained latitude is the assumed latitude used in the prime vertical method to determine the longitude. Dead reckoning is the navigational term for calculating one's new position from the previous position, course and distance (calculated from the vessel's average speed and time elapsed).

Prime vertical method makes use of the altitude when a celestial body is on the prime vertical to determine longitude. The celestial body is said to be on the prime vertical when its azimuth is 90° or 270°. The procedures of the method are described below.

(1) Obtain an assumed latitude, **$\mathbf{Lat_{AP}}$** and the declination of the body, **dec**.

(2) Use Lat_{AP} and Dec to calculate the observed altitude, **Ho**, of the body.

$$Ho = \sin^{-1}\left[\frac{\sin(Dec)}{\sin(Lat_{AP})}\right]$$

(3) Obtain sextant altitude, **Hs**, by adding navigational errors to Ho.

(4) Note the **GMT** when the sextant reading gives Hs.

(5) From the Nautical Almanac, for the GMT, obtain the **GHA** of the body.

(6) Use Ho and Dec to calculate the meridian angle, **t**, east or west of the body.

$$t = \sin^{-1}\left[\frac{\cos(Ho)}{\cos(Dec)}\right]$$

(7) Use GHA and t to determine longitude.

$$Lon = GHA \pm t$$

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